Problem 1.52

For Theorem 2, show that (d) ⇒ (a), (a) ⇒ (c), (c) ⇒ (b), (b) ⇒ (c), and (c) ⇒ (a).

Solution

Theorem 2 says that the following conditions are equivalent.

(a) $\nabla \cdot \mathbf{F} = 0$ everywhere.

(b) $\iint \mathbf{F} \cdot d\mathbf{S}$ is independent of surface for any given boundary line.

(c) $\oiint \mathbf{F} \cdot d\mathbf{S} = 0$ for any closed surface.

(d) $\mathbf{F}$ is the curl of some vector function: $\mathbf{F} = \nabla \times \mathbf{A}$.

(d) ⇒ (a)

Assume that $\mathbf{F}$ is the curl of some vector function: $\mathbf{F} = \nabla \times \mathbf{A}$. Show that $\nabla \cdot \mathbf{F} = 0$.

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{A}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k A_k\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) \frac{\partial A_k}{\partial x_j}\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} \frac{\partial A_k}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{il} \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jki} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3} \varepsilon_{jik} \frac{\partial^2 A_k}{\partial x_j \partial x_i} \qquad \text{(let } i \text{ be } j \text{ and let } j \text{ be } i.) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jik} \frac{\partial^2 A_k}{\partial x_j \partial x_i} \qquad \text{(limits are constant, so interchange sums)} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jik} \frac{\partial^2 A_k}{\partial x_i \partial x_j} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} (-\varepsilon_{ijk}) \frac{\partial^2 A_k}{\partial x_i \partial x_j} = -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial^2 A_k}{\partial x_i \partial x_j} = 0
\end{aligned}$$

(a) ⇒ (c)

Assume that $\nabla \cdot \mathbf{F} = 0$ everywhere and show that $\oiint \mathbf{F} \cdot d\mathbf{S} = 0$ for any closed surface.

$$\nabla \cdot \mathbf{F} = 0$$

Integrate both sides over any volume D with surface, bdy D.

$$\iiint_D (\nabla \cdot \mathbf{F})\, dV = \iiint_D (0)\, dV$$

Use Gauss's theorem on the left and evaluate the integral on the right.

$$\oiint_{\text{bdy } D} \mathbf{F} \cdot d\mathbf{S} = 0$$

(c) ⇒ (b)

Assume that

$$\oiint_{\text{bdy } D} \mathbf{F} \cdot d\mathbf{S} = 0$$

for any closed surface, bdy D. Use Gauss's theorem to turn this surface integral into a volume integral over the enclosed volume D.

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = 0$$

Since this holds for any volume, $\nabla \cdot \mathbf{F} = 0$ everywhere. And that means there exists a vector potential function $\mathbf{A}$ such that $\mathbf{F} = \nabla \times \mathbf{A}$. Consider the integral of $\mathbf{F}$ over an arbitrary open surface S with boundary line, bdy S, and use Stokes's theorem.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{A} \cdot d\mathbf{l}$$

Regardless of what S is, the surface integral is always equal to a line integral over the boundary.

(b) ⇒ (c)

Assume that

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

is independent of surface S for any given boundary line, bdy S. This implies that $\mathbf{F}$ is the curl of a vector potential function: $\mathbf{F} = \nabla \times \mathbf{A}$. By Stokes's theorem, then,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{A} \cdot d\mathbf{l},$$

but this is not important. What is important is the closed surface integral of $\mathbf{F}$.

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \oiint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{A} \cdot d\mathbf{l} = \int_{\mathbf{a}}^{\mathbf{a}} \mathbf{A} \cdot d\mathbf{l} = 0$$

It's zero because the boundary line of a closed surface is a single point.

(c) ⇒ (a)

Assume that

$$\oiint_{\text{bdy } D} \mathbf{F} \cdot d\mathbf{S} = 0$$

for any closed surface, bdy D. Apply Gauss's theorem to turn this surface integral into a volume integral.

$$\iiint_D (\nabla \cdot \mathbf{F})\, dV = 0$$

Since this is true for any volume D, $\nabla \cdot \mathbf{F} = 0$ everywhere.